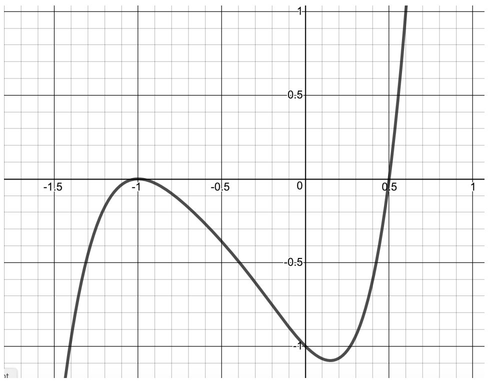
MATH 1650: COMPLEX ZEROS OF POLYNOMIALS

EXAMPLE: Let $p(x) = 2x^5 + 5x^4 + 5x^3 + 2x^2 - x - 1$.

- Since p is a fifth degree polynomial, we know that we need to perform at least three successful divisions to get the quotient down to a quadratic function. Using desmos, we create the graph below on the left. The RZT suggests $x = -1$ and $x = \frac{1}{2}$ as zeros, which we verify using synthetic division below on the right.



$$\begin{array}{r|rrrrrr}
-1 & 2 & 5 & 5 & 2 & -1 & -1 \\
 & \downarrow & -2 & -3 & -2 & 0 & 1 \\
\hline
-1 & 2 & 3 & 2 & 0 & -1 & \boxed{0} \\
 & \downarrow & -2 & -1 & -1 & 1 & \\
\hline
\frac{1}{2} & 2 & 1 & 1 & -1 & \boxed{0} & \\
 & \downarrow & 1 & 1 & 1 & & \\
\hline
 & 2 & 2 & 2 & \boxed{0} & &
\end{array}$$

Our quotient is $2x^2 + 2x + 2$. To solve $2x^2 + 2x + 2 = 0$, we first factor: $2(x^2 + x + 1) = 0$ and set $x^2 + x + 1 = 0$. Since this quadratic does not factor nicely, we appeal to the quadratic formula. We check the discriminant: $b^2 - 4ac = (1)^2 - 4(1)(1) = -3 < 0$ which means the remaining zeros of f are non-real. Hence, we use our synthetic division tableau to factor $f(x)$ over the real numbers as:

$$f(x) = (x+1)^2 \left(x - \frac{1}{2}\right)\left(2x^2 + 2x + 2\right) = 2\,(x+1)^2 \left(x - \frac{1}{2}\right)\left(x^2 + x + 1\right)$$

- Find the non-real zeros of f and factor $f(x)$ over the complex numbers.

Using the Quadratic Formula, we solve $x^2 + x + 1 = 0$: $x = \dfrac{1 \pm \sqrt{1 - 4(1)(1)}}{2(1)} = \dfrac{1 \pm \sqrt{-3}}{2} = \dfrac{1 \pm i\sqrt{3}}{2}$.

Applying the Complex Factorization Theorem we factor $f(x)$ as:

$$f(x) = 2\,(x+1)^2 \left(x - \frac{1}{2}\right)\left(x - \left[\frac{1 + i\sqrt{3}}{2}\right]\right)\left(x - \left[\frac{1 - i\sqrt{3}}{2}\right]\right)$$

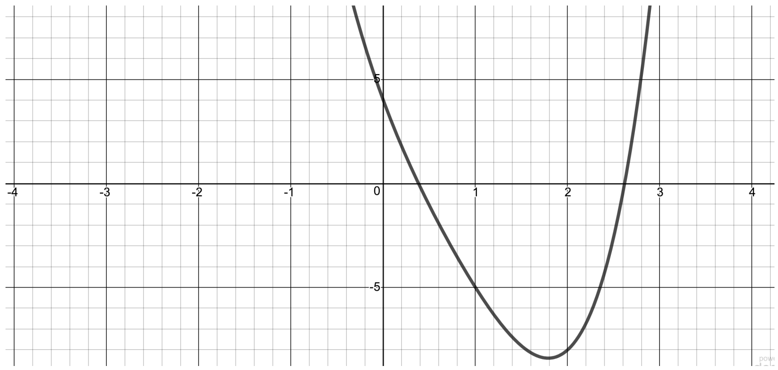
EXAMPLE: Let $p(z) = z^4 - 3z^3 + 5z^2 - 12z + 4$.

- Show p has no rational zeros.

  According to the RZT, the only possible rational zeros of p are $\pm\dfrac{\text{factors of } 4}{\text{factors of } 1} = \pm\dfrac{1, 2, 4}{1}$.

  Hence, our list of possible rational zeros as $\{\pm 1, \pm 2, \pm 4\}$.

  A quick look at the graph reveals neither of them are on the list of possible rational zeros.



- Use synthetic division to show $z = 2i$ is a zero of p and use this to find the remaining complex zeros of p.

  HINT: Once you show $z = 2i$ is a zero, what else must be a zero?

  If $z = 2i$ is a zero, then $z = -2i$ is also a zero. Hence, we make two divisions below:

$$\begin{array}{r|rrrrr}
2i & 1 & -3 & 5 & -12 & 4 \\
 & \downarrow & 2i & -4-6i & 12+2i & -4 \\
\hline
-2i & 1 & -3+2i & 1-6i & 2i & \boxed{0} \\
 & \downarrow & -2i & 6i & -2i & \\
\hline
 & 1 & -3 & 1 & \boxed{0} &
\end{array}$$

  Our quotient polynomial is $z^2 - 3z + 1$. Solving $z^2 - 3z + 1 = 0$ using the Quadratic Formula gives:

$$z = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

  Hence, the complex zeros of p are $z = \pm 2i$ and $z = \dfrac{3 \pm \sqrt{5}}{2}$.

- Factor $p(z)$ over the complex numbers.

  Using the complex factorization theorem, we get: $p(z) = (z - 2i)(z + 2i)\left(z - \left[\dfrac{3+\sqrt{5}}{2}\right]\right)\left(z - \left[\dfrac{3-\sqrt{5}}{2}\right]\right)$.

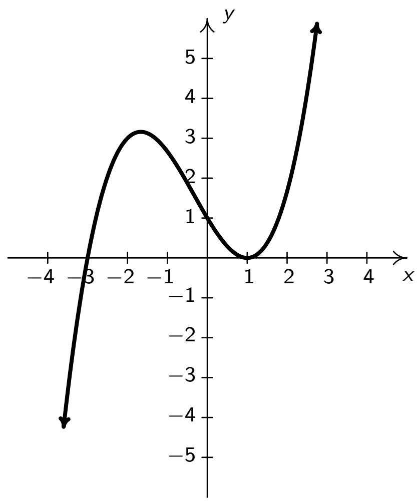
- Factor $p(z)$ over the real numbers.

  To factor $p(z)$ over the real numbers, we need to eliminate any non-real coefficients in our factorization. So we first multiply: $(z - 2i)(z + 2i) = z^2 - 4i^2 = z^2 + 4$. Hence, over the real numbers,

$$p(z) = (z^2 + 4)\left(z - \left[\frac{3+\sqrt{5}}{2}\right]\right)\left(z - \left[\frac{3-\sqrt{5}}{2}\right]\right)$$

EXAMPLE: Find a possible formula for the polynomial functions described below:

- the polynomial function graphed below:



$y = f(x)$

From the graph, it looks like $x = -3$ is a zero of multiplicity 1 and $x = 1$ is a zero of multiplicity 2. From the complex factorization theorem, this means $f(x) = a(x+3)(x-1)^2$.

To find a, we note $(0,1)$ is on the graph of $y = f(x)$ which means $f(0) = 1$.

Since $f(0) = a(0+3)(0-1)^2 = 3a$, we get $3a = 1$ so $a = \frac{1}{3}$.

Hence, a possible formula for $f(x)$ is $f(x) = \frac{1}{3}(x+3)(x-1)^2$.

- $p(t)$ where:

  - p has real number coefficients.
  - p is degree 4.
  - $t = 2 - i$ is zero.
  - the point $(-3, 0)$ is a local maximum.

Since p has real number coefficients, if $t = 2 - i$ is a zero, then so is $t = 2 + i$.

If $(-3,0)$ is a local maximum, the graph of $y = p(t)$ touches and rebounds at $(-3, 0)$.

Hence, $t = -3$ is a zero of even multiplicity.

Hence, the complex factorization theorem gives: $p(t) = a(t - [2-i])(t - [2+i])(t+3)^2$

In order for $(-3,0)$ to be a local **maximum**, the coefficient $a < 0$.

Since we have no other information, $p(t) = a(t - [2-i])(t - [2+i])(t+3)^2$ for any choice of $a < 0$.